Introducing e^x

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The exponential function $y = e^x$ is the great creation of calculus. Algebra is all we need for $x, x^2, \ldots, x^n$. Trigonometry leads us to $\sin x$ and $\cos x$. But the last in this short list of all-important functions cannot come so directly. That is because e^x requires us, at one point or another, to take a *limit*. The most important function of calculus depends on the central idea of the whole subject: perfect for every teacher.

Still a very big question remains. *How do we approach* e^x? That limiting step can come in many places, sometimes openly and sometimes hidden. At the end of this note we mention several of these approaches (the reader may know others). My chief purpose in this paper is to advocate the choice that seems most direct and straightforward. This choice builds on what we know (the derivative of x^n), it goes immediately to the properties we use, and it brings out the central goal of calculus: to connect functions with their rates of change.

What we know:	The derivative of cx^n is ncx^{n-1}
Property we use:	The product of e^x and e^X is e^{x+X}
Connection we need:	***The derivative of*** e^x ***is*** e^x

Calculus is about *pairs of functions*. Function **1** (the distance we travel or the height we climb) is changing. Function **2** (the velocity df/dt or the slope dy/dx) tells the rate of change. From one of those functions, we find the other.

This is the heart of calculus, and we must not let students lose sight of it. The relation of Function **1** to Function **2** is learned by examples more than by definitions, and those great functions are the right ones to remember:

$$y = x^n \qquad y = \sin x \text{ and } \cos x \qquad y = e^x \text{ and } e^{cx}$$

With e^x as our goal, let me suggest that we go straight there. If we hide its best property, students won't find it (and won't feel it). What makes this function special?

The slope of e^x is e^x

Function **1** equals Function **2**

$\boldsymbol{y = e^x}$ **solves the differential equation** $\boldsymbol{dy/dx = y}$**.**

Differential equations are laws of change. The whole purpose of calculus is to understand change. It is wonderful to see the most important differential equation so early, and doubly wonderful to solve it.

One more requirement will eliminate solutions like $y = 2e^x$ and $y = 8e^x$ (the 2 and 8 will appear on both sides of $dy/dx = y$, so the equation still holds). At $x = 0$, e^0 will be the "zeroth power" of the positive number e. *All zeroth powers are* 1. So we want $y = e^x$ to equal 1 when $x = 0$:

$$\boldsymbol{y = e^x \textit{ is the solution of } \frac{dy}{dx} = y \textit{ that starts from } y = 1 \textit{ at } x = 0.}$$

Before that solution, draw what it means to have $y = dy/dx$. The slope at $x = 0$ must be $dy/dx = 1$ (since $y = 1$). So the curve starts upward, along the line $y = 1 + x$. But as y increases, its slope increases. So the graph goes up faster (and then faster). "Exponential growth" means that the function and its slope stay proportional.

The time you give to that graph is well spent. Once formulas arrive, they tend to take over. The formulas are exactly right, and the graph is only approximately right. But the graph also shows $e^{-x} = 1/e^x$, rapidly approaching but never touching $y = 0$.

This introduction ends here, before e^x is formally presented. But a wise reader knows that we all pay closer attention when we are convinced that a new person or a new function is important. I hope you will allow me to present e^x partly as if to a class, and partly as a suggestion to all of us who teach calculus.

Constructing $y = e^x$

I will solve $dy/dx = y$ a step at a time. At the start, $y = 1$ means that $dy/dx = 1$:

$$\textbf{Start} \quad \begin{array}{r} y = 1 \\ dy/dx = 1 \end{array} \qquad \textbf{Change } \boldsymbol{y} \quad \begin{array}{r} y = 1 + x \\ dy/dx = 1 \end{array} \qquad \textbf{Change } \boldsymbol{\frac{dy}{dx}} \quad \begin{array}{r} y = 1 + x \\ dy/dx = 1 + x \end{array}$$

After the first change, $y = 1 + x$ has the correct derivative $dy/dx = 1$. But then I had to change dy/dx to keep it equal to y. And I can't stop there:

$$\begin{array}{ccccccccc} y & 1 & & 1+x & & 1+x+\frac{1}{2}x^2 & & \text{cubic} \\ \text{equals} & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ dy/dx & 1 & & 1+x & & 1+x+\frac{1}{2}x^2 & & \text{cubic} \end{array}$$

The extra $\frac{1}{2}x^2$ gives the correct x in the slope. Then $\frac{1}{2}x^2$ also has to go into dy/dx, to keep it equal to y. Now we need a new term with this derivative $\frac{1}{2}x^2$.

The term that gives $\frac{1}{2}x^2$ has x^3 divided by 6. The derivative of x^n is nx^{n-1}, so I must divide by n (to cancel correctly). Then the derivative of $x^3/6$ is $3x^2/6 = \frac{1}{2}x^2$ as we wanted. After that comes x^4 divided by 24:

$$\begin{array}{lcl} \dfrac{x^3}{6} = \dfrac{x^3}{(3)(2)(1)} & \text{has slope} & \dfrac{x^2}{(2)(1)} \\ \dfrac{x^4}{24} = \dfrac{x^4}{(4)(3)(2)(1)} & \text{has slope} & \dfrac{4x^3}{(4)(3)(2)(1)} = \dfrac{x^3}{6}. \end{array}$$

The pattern becomes more clear. The x^n term is divided by n *factorial*, which is $n! = (n)(n-1)\ldots(1)$. The first five factorials are $1, 2, 6, 24, 120$. **The derivative of that term** $x^n/n!$ **is the previous term** $x^{n-1}/(n-1)!$ (because the n's cancel). As long as we don't stop, this sum of infinitely many terms does achieve $dy/dx = y$:

$$\boldsymbol{y(x) = e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots + \tfrac{1}{n!}x^n + \cdots} \tag{1}$$

Here is the function. Take the derivative of every term and this series appears again.

If we substitute $x = 10$ into this series, do the infinitely many terms add to a finite number e^{10}? *Yes*. The numbers $n!$ grow much faster than 10^n (or any other x^n). So the terms $10^n/n!$ in this "exponential series" become extremely small as $n \to \infty$. Analysis shows that the sum of the series (which is $y = e^x$) does achieve $dy/dx = y$.

Note 1 The *geometric series* $1 + x + x^2 + x^3 + \cdots$ adds up to $1/(1-x)$. This is the most important series in mathematics, but it runs into a problem at $x = 1$: the sum $1 + 1 + 1 + 1 + \cdots$ is infinite. The series for e^x is entirely different, because the powers x^n are divided by the rapidly growing numbers $n! = n$ *factorial*.

Every term $x^n/n!$ is the previous term multiplied by x/n. Those multipliers approach zero and the limit step succeeds (the infinite series has a finite sum). This is a great example to meet, long before you learn more about convergence and divergence.

Note 2 Here is another way to look at that series for e^x. Start with x^n and take its derivative n times. First get nx^{n-1} and then $n(n-1)x^{n-2}$. Finally the nth derivative is $n(n-1)(n-2)\ldots(1)x^0$, which is n *factorial*. When we divide by that number, **the nth derivative of $x^n/n!$ is equal to 1**. All other derivatives are zero at $x = 0$.

Now look at e^x. All its derivatives are still e^x, so they also equal 1 at $x = 0$. *The series is matching every derivative of e^x at the starting point $x = 0$.*

Note 3 ***Set $x = 1$ in the exponential series***. This tells us the amazing number $e^1 = e$:

$$\boldsymbol{e = 1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \tfrac{1}{120} + \cdots = 2.71828\ldots} \tag{2}$$

The first three terms add to 2.5. The first five terms almost reach 2.71. *We never reach* 2.72. It is certain that e is not a fraction. It never appears in algebra, but it is the key number for calculus.

Multiplying by Adding Exponents

Is it true that e times e equals e^2? Up to now, e and e^2 come separately. We substitute $x = 1$ and then $x = 2$ in the infinite series. The wonderful fact is that for every x, the series produces the "xth power of the number e." When $x = -1$, we get e^{-1} which is $1/e$:

Set $x = -1$ $$e^{-1} = \frac{1}{e} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \cdots$$

If we multiply that series for $1/e$ by the series for e, we get 1.

The best way is to go straight for all multiplications of e^x times any power e^X. The rule of adding exponents says that the answer is e^{x+X}. The series must say this too. When $x = 1$ and $X = -1$, this rule produces e^0 from e^1 times e^{-1}.

Add the exponents $$\boldsymbol{(e^x)(e^X) = e^{x+X}} \tag{3}$$

We only know e^x and e^X from the infinite series. For this all-important rule, we can multiply those series and recognize the answer as the series for e^{x+X}. Make a start:

Multiply each term	$e^x = \mathbf{1} + \boldsymbol{x} + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$	
$\boldsymbol{e^x}$ **times** $\boldsymbol{e^X}$	$e^X = \mathbf{1} + \boldsymbol{X} + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \cdots$	
Hoping for $\boldsymbol{e^{x+X}}$	$(e^x)(e^X) = \mathbf{1} + \boldsymbol{x} + \boldsymbol{X} \quad + \frac{1}{2}x^2 + \boldsymbol{xX} + \frac{1}{2}X^2 + \cdots$	(4)

$1 + x + X$ is the right start for e^{x+X}. Then comes $\frac{1}{2}(x+X)^2$:

$$\frac{1}{2}(x+X)^2 = \frac{1}{2}(x^2 + 2xX + X^2) \quad \text{matches the “second degree” terms in (4).}$$

The step to third degree takes a little longer, but it also succeeds:

$$\frac{1}{6}(x+X)^3 = \frac{1}{6}x^3 + \frac{3}{6}x^2X + \frac{3}{6}xX^2 + \frac{1}{6}X^3 \quad \text{matches the next terms in (4).}$$

For high powers of $x + X$ we need the *binomial theorem* (or a healthy trust that mathematics comes out right). When e^x multiplies e^X, this produces all the products of $(x^n/n!)$ times $(X^m/m!)$. Now look for that same term inside the series for e^{x+X}:

$$\text{Inside } \frac{(x+X)^{n+m}}{(n+m)!} \text{ is } \left(\frac{x^n X^m}{(n+m)!}\right) \text{ times } \left(\frac{\boldsymbol{(n+m)!}}{\boldsymbol{n!\,m!}}\right) \text{ which gives } \frac{x^n X^m}{n!\,m!}. \tag{5}$$

That binomial number $\boldsymbol{(n+m)!/n!\,m!}$ counts the number of ways to choose n aces out of $n+m$ aces. Out of 4 aces, you could choose 2 aces in $4!/2!2! = 6$ ways. There are 6 ways to choose 2 x's out of $xxxx$. This number 6 will be the coefficient of x^2X^2 in $(x+X)^4$.

In the fourth degree term, that 6 is divided by 4! (to produce $1/4$). When e^x multiplies e^X, $\frac{1}{2}x^2$ multiplies $\frac{1}{2}X^2$ (which also produces $1/4$). All terms are correct, but we are not going there—we accept $(e^x)(e^X) = e^{x+X}$ as now confirmed.

Second proof A different way to see this rule for $(e^x)(e^X)$ is based on $dy/dx = y$. Start from $y = 1$ at $x = 0$. At the point x, you reach $y = e^x$. Now go an additional distance X to arrive at e^{x+X}.

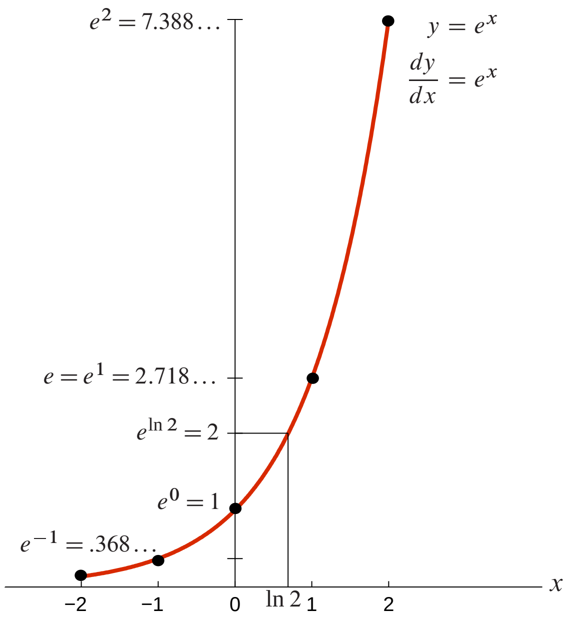
Notice that the additional part starts from e^x (instead of starting from 1). That starting value e^x will multiply e^X in the additional part. So e^x times e^X must be the same as e^{x+X}. This is a “differential equations proof” that the exponents are added. (Personally, I am happy to multiply the series and match the terms.)

The rule immediately gives e^x times e^x. The answer is $e^{x+x} = e^{2x}$. If we multiply again by e^x, we find $(e^x)^3$. This is equal to $e^{2x+x} = e^{3x}$. We are finding a rule for all powers $(e^x)^n = (e^x)(e^x)\cdots(e^x)$:

Multiply exponents $$\boldsymbol{(e^x)^n = e^{nx}} \tag{6}$$

This is easy to see for $n = 1, 2, 3, \ldots$ and then $n = -1, -2, -3, \ldots$ *It remains true for all numbers x and n.*

That last sentence about "all numbers" is important! Calculus cannot develop properly without working with all exponents (not just whole numbers or fractions). The infinite series (1) defines e^x for every x and we are on our way. Here is the graph that shows **Function (1) = Function (2) =** $\boldsymbol{e^x = \exp(x)}$**.**



$$\boldsymbol{(e^x)(e^X) = e^{x+X}}$$
$$\boldsymbol{(e^x)^n = e^{nx}}$$
$$\boldsymbol{e^{\ln y} = y}$$

The Exponentials $\boldsymbol{2^x}$ and $\boldsymbol{b^x}$

We know that $2^3 = 8$ and $2^4 = 16$. But what is the meaning of 2^π? One way to get close to that number is to replace π by 3.14 which is 314/100. As long as we have a fraction in the exponent, we can live without calculus:

Fractional power $\qquad 2^{314/100} =$ 314th power of the 100th root $2^{1/100}$.

But this is only "close" to 2^π. And in calculus, we will want the exact slope of the curve $y = 2^x$. The good way is to connect 2^x with e^x, whose slope we know (it is e^x again). So we need to connect 2 with e.

The key number is the **logarithm of 2**. This is written "ln 2" and it is the power of e that produces 2. It is specially marked on the graph of e^x:

Natural logarithm of 2 $\qquad e^{\ln 2} = 2$

This number ln 2 is about 7/10. A calculator knows it with much higher accuracy. In the graph of $y = e^x$, the number ln 2 on the x axis produces $y = 2$ on the y axis.

This is an example where we want the output $y = 2$ and we ask for the input $x = \ln 2$. That is the opposite of knowing x and asking for y. "The logarithm $x = \ln\, y$ is the *inverse* of the exponential $y = e^x$." This idea is explained in two video lectures on **ocw.mit.edu**—inverse functions are not always simple.

When we have the number $\ln 2$, meeting the requirement $2 = e^{\ln 2}$, we can take the xth power of both sides:

$$\textbf{Powers of 2 from powers of } \boldsymbol{e} \qquad 2 = e^{\ln 2} \quad \text{and} \quad \mathbf{2^x = e^{x \ln 2}}. \tag{7}$$

All powers of e are defined by the infinite series. The new function 2^x also grows exponentially, but not as fast as e^x (because 2 is smaller than e). Probably $y = 2^x$ could have the same graph as e^x, if I stretched out the x axis. That stretching multiplies the slope by the constant factor $\ln 2$. Here is the algebra:

$$\textbf{Slope of } \boldsymbol{y = 2^x} \qquad \frac{d}{dx}2^x = \frac{d}{dx}e^{x \ln 2} = (\ln 2)\, e^{x \ln 2} = \mathbf{(ln\ 2)2^x}.$$

For any positive number b, the same approach leads to the function $y = b^x$. First, find the natural logarithm $\ln b$. This is the number (positive or negative) so that $b = e^{\ln b}$. Then take the xth power of both sides:

$$\textbf{Connect } \boldsymbol{b} \textbf{ to } \boldsymbol{e} \qquad b = e^{\ln b} \ \text{ and } \ b^x = e^{x \ln b} \ \text{ and } \ \frac{d}{dx}b^x = (\ln b)b^x \tag{8}$$

When b is e (the perfect choice), $\ln b = \ln e = 1$. When b is e^n, then $\ln b = \ln e^n = n$. "***The logarithm is the exponent***." Thanks to the series that defines e^x for every x, that exponent can be any number at all.

Allow me to mention Euler's Great Formula $\boldsymbol{e^{ix} = \cos x + i \sin x}$. The exponent ix has become an **imaginary number**. (You know that $i^2 = -1$.) If we faithfully use $\cos x + i \sin x$ at 90° and 180° (where $x = \pi/2$ and $x = \pi$), we arrive at these wonderful facts:

$$\textbf{Imaginary exponents} \qquad \boldsymbol{e^{i\pi/2} = i} \quad \text{and} \quad \boldsymbol{e^{i\pi} = -1}. \tag{9}$$

Those equations are not imaginary, they come from the great series for e^x.

Continuous Compounding of Interest

There is a different and important way to reach e and e^x (not by an infinite series). We solve the key equation $dy/dx = y$ in small steps. As these steps approach zero (a limit is always involved !) the small-step solution Y becomes the exact $y = e^x$.

I can explain this idea in two different languages. Each step multiplies Y by $1 + \Delta x$:

1. *Compound interest*. After each step Δx, the interest is added to Y. Then the next step begins with a larger amount $(1 + \Delta x)Y$.
2. *Finite differences*. The continuous dy/dx is replaced by small steps $\Delta Y/\Delta x$:

$$\frac{dy}{dx} = y \ \text{ changes to } \ \frac{Y(x + \Delta x) - Y(x)}{\Delta x} = Y(x) \ \text{ still with } Y(0) = 1. \tag{10}$$

Let me compute compound interest when 1 year is divided into 12 months. The interest rate is 100% and you start with $Y(0)=\$1$. If you only got interest once, at the end of the year, then you have $Y(1)=\$2$.

If interest is added every month, you now get $\frac{1}{12}$ of 100% each time (12 times). So Y is multiplied each month by $1+\frac{1}{12}$. (The bank adds $\frac{1}{12}$ for every 1 you have.) Do this 12 times and the final value \$2 is improved to \$2.61:

$$\textbf{After 12 months} \qquad Y(1)=\left(1+\frac{1}{12}\right)^{12}=\$2.61$$

Now add interest every day. $Y(0)=\$1$ is multiplied 365 times by $1+\frac{1}{365}$:

$$\textbf{After 365 days} \qquad Y(1)=\left(1+\frac{1}{365}\right)^{365}=\$2.71 \ (\textit{close to } e)$$

Very few banks use minutes, and nobody divides the year into $N=31,536,000$ seconds. It would add less than a penny to \$2.71. But many banks are willing to use *continuous compounding*, the limit as $N\to\infty$. After one year you have \$$e$:

$$\textbf{Another limit gives } \boldsymbol{e} \qquad \left(1+\frac{1}{N}\right)^{N}\to e=2.718\ldots \text{ as } N\to\infty \tag{11}$$

This is the same number e as $1+1+\frac{1}{2}+\frac{1}{6}+\cdots$ from the approach that I prefer. To match this continuous compounding with e^x, invest at the 100% rate for x years. Now each of the N steps is x/N years. Again the bank multiplies at every step by $1+\frac{x}{N}$. The 1 keeps what you have, the x/N adds the interest in that step. After N steps you are close to e^x:

$$\textbf{Another formula for } \boldsymbol{e^x} \qquad \left(1+\frac{x}{N}\right)^{N}\to e^x \text{ as } N\to\infty \tag{12}$$

Comment. I would allow this second approach into my classroom, since everything about e^x is so important. But I wouldn't prove that it gives the same e^x as the equation $dy/dx=y$. Of course this is quite reasonable, since the derivative of $(1+\frac{x}{N})^N$ is $(1+\frac{x}{N})^{N-1}$. And equally reasonable to expect the difference equation $\Delta Y/\Delta x=Y$ to stay close to $dy/dx=y$.

Hairer and Wanner [4] have compared the product $\left(1+\frac{1}{N}\right)^N$ to the partial sum $1+\cdots+1/N!$ of the series:

$N=1$	2.000	2
2	2.250	2.5
3	2.370	2.67
4	2.441	2.708
5	2.488	2.7166
6	2.522	2.71805
7	2.546	2.718253
8	2.566	2.7182787
9	2.581	2.71828152
10	2.594	2.718281801
11	2.581	2.7182818261
12	2.594	2.71828182828

One column shows the slow convergence of the discrete $\Delta Y/\Delta x = Y$ to the continuous $dy/dx = y$. The error $y-Y$ is of order $\Delta x = 1/N$. (This "Euler method" is still chosen for difficult problems.) The other column has errors of order $1/N!$, more like a modern "spectral method."

Euler himself had seen this contrast before 1748, the date of his great textbook [2]. Johann Bernoulli connected logarithms to exponential series in 1697 [1]. And by 1751, Euler could resolve a hot debate between Bernoulli and Leibniz about the logarithm of a negative number [3]. The key was his wonderful formula $e^{ix} = \cos x + i\,\sin x$.

Third approach Authors frequently produce e^x by starting with 2^x and 3^x. Those curves have slopes proportional to 2^x and 3^x. The slope of any function b^x is proportional to that function:

$$\text{slope} = \text{limit of } \frac{b^{x+h}-b^x}{h} = b^x \text{ times } \left(\text{limit of } \frac{b^h-1}{h}\right) = Cb^x. \tag{13}$$

That number C is smaller than 1 for $b=2$, and larger than 1 for $b=3$. Somewhere between, there must be a number for which $C=1$. This reasoning produces a number e for which the slope of e^x is e^x.

It is not right to criticize this approach on mathematical grounds. Pedagogically, I don't see how a student can build on it. To me, the steps from 1 to $1+x$ to $1+x+\frac{1}{2}x^2$ are going somewhere. We are seeing central ideas of calculus, the tangent line $y=1+x$ that gives linear approximation and the tangent parabola that gives quadratic approximation. The motivation is clear and the correctness can be seen term by term, by using (and reinforcing) the derivative of x^n.

An infinite series is still a big jump. But it is good to show students where we are going, by an example that we really need and use.

The Equation $dy/dx = ay$

The "use" of calculus is to understand change. The first step is from y to dy/dx (Function **1** to Function **2**). The next step reaches d^2y/dx^2 and its meaning and importance (this can be Function **3**). There is one more absolutely crucial step, to connect those functions by equations like $dy/dx = y$ and $d^2y/dx^2 = -y$. These are fundamental equations of nature and why wouldn't we solve them?

Yes, nonlinear problems can wait for that future course on differential equations. But the essential points are clearest for three linear equations with constant coefficients:

$$\frac{dy}{dx} = y \qquad \frac{dy}{dx} = ay \qquad \frac{dy}{dx} = ay + s.$$

The solution to the first also solves the second, after a scale change on the x axis:

$$\textbf{Change the interest rate to } \boldsymbol{a} \qquad \boldsymbol{\frac{dy}{dx} = ay} \ \textbf{\textit{is solved by}} \ \boldsymbol{y(x) = e^{ax}} \tag{14}$$

The series for e^{ax} is $1 + ax + \frac{1}{2}(ax)^2 + \cdots$ and we take its derivative:

$$\frac{d}{dx}(e^{ax}) = a + a^2x + \cdots = a(1 + ax + \cdots) = \boldsymbol{ae^{ax}} \tag{15}$$

The derivative of e^{ax} brings down the extra factor a. So $y = e^{ax}$ solves $dy/dx = ay$.

This soon becomes a key example of the chain rule. And the third equation has a constant solution $-s/a$ to add to the exponentials Ce^{ax}.

Fourth Approach by Inverse Functions

Instead of constructing $y = e^x$, we could construct the inverse function $x = \ln y$. Either way will yield all pairs (x, y), and the natural logarithm needs only an ordinary integration:

$$\text{Invert } \frac{dy}{dx} = y \text{ to } \frac{dx}{dy} = \frac{1}{y} \text{ and then } x = \int \frac{1}{y}\,dy. \tag{16}$$

Starting that integration at $y = 1$ gives the correct value $x = \ln 1 = 0$. After inversion this is $y = e^{0} = 1$. And introducing t as a dummy variable leaves $x = \ln y = \int_1^y \frac{1}{t}\,dt$.

Now the "limiting step" that e^x always needs is in the definition of the integral. The key property $e^x e^X = e^{x+X}$ becomes $\ln(yY) = \ln y + \ln Y$. This is proved directly from the integral.

This fourth approach has its attractions. But look for the ideas that need to be understood first:

1. The meaning of an inverse function
2. The definition of an integral
3. The chain rule for $x = f^{-1}(y)$ that gave $(dx/dy)(dy/dx) = 1$.

Maybe there is a way to escape that chain rule, but not the others. So e^x would have to come long after the derivative of x^n. "Early Transcendentals" will be impossible this way, and the ideas themselves seem much more subtle.

Explicit constructions are the winners – you can say "here is the function."

A Personal Note

When I wrote a textbook on calculus twenty years ago, I didn't appreciate e^x. Of course it was sure to be important. But the exponential wasn't seen as the organizing function for differential calculus.

It was in preparing video lectures on "Highlights of Calculus" that e^x moved into its right place. Without all the details of a complete course, those videos are open to everyone on MIT's OpenCourseWare (*ocw.mit.edu*, Highlights for High School). They led to a new edition of *Calculus* and to this paper.

I hope you will feel that a stronger emphasis on e^x is right. And I hope that this suggested presentation uses what students know, to go directly toward this wonderful function at the heart of calculus and its applications.

References

1. Joh. Bernoulli, Principia calculi exponentialium, Opera I (1697) 179–187.

2. L.Euler, *Introductio in analysin infinitorum*, Opera **8**, (1748).

3. L.Euler, De la contreverse entre Mrs. Leibniz and Bernoulli sur les logarithmes des nombres négatifs et imaginaires, Mém. Acad. Sc. Berlin **5** (1751) 139–179; Opera **17** 195–232. (Note that Mrs.= Messieurs)

4. E. Hairer and G. Wanner, *Analysis by Its History*, Springer (2008).