4

ORTHOGONALITY

ORTHOGONALITY OF THE FOUR SUBSPACES ■ 4.1

Two vectors are orthogonal when their dot product is zero: $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ or $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0$. This chapter moves up a level, from orthogonal vectors to ***orthogonal subspaces***. Orthogonal means the same as perpendicular.

Subspaces entered Chapter 3 with a specific purpose—to throw light on $A\boldsymbol{x} = \boldsymbol{b}$. Right away we needed the column space (for $\boldsymbol{b}$) and the nullspace (for $\boldsymbol{x}$). Then the light turned onto A^{T}, uncovering two more subspaces. Those four fundamental subspaces reveal what a matrix really does.

A matrix multiplies a vector: *A times* $\boldsymbol{x}$. At the first level this is only numbers. At the second level $A\boldsymbol{x}$ is a combination of column vectors. The third level shows subspaces. But I don't think you have seen the whole picture until you study Figure 4.1. It fits the subspaces together, to show the hidden reality of A times $\boldsymbol{x}$. The 90° angles between subspaces are new—and we have to say what they mean.

The row space is perpendicular to the nullspace. Every row of A is perpendicular to every solution of $A\boldsymbol{x} = \boldsymbol{0}$. That gives the 90° angle on the left side of the figure. This perpendicularity of subspaces is Part 2 of the Fundamental Theorem of Linear Algebra.

May we add a word about the left nullspace? It is never reached by $A\boldsymbol{x}$, so it might seem useless. But when $\boldsymbol{b}$ is outside the column space—when we want to solve $A\boldsymbol{x} = \boldsymbol{b}$ and can't do it—then this nullspace of A^{T} comes into its own. It contains the error in the "least-squares" solution. That is the key application of linear algebra in this chapter.

Part 1 of the Fundamental Theorem gave the dimensions of the subspaces. The row and column spaces have the same dimension r (they are drawn the same size). The two nullspaces have the remaining dimensions $n - r$ and $m - r$. Now we will show that ***the row space and nullspace are orthogonal subspaces inside*** $\mathbf{R}^n$.

DEFINITION Two subspaces $\boldsymbol{V}$ and $\boldsymbol{W}$ of a vector space are ***orthogonal*** if every vector $\boldsymbol{v}$ in $\boldsymbol{V}$ is perpendicular to every vector $\boldsymbol{w}$ in $\boldsymbol{W}$:

$$\boldsymbol{v} \cdot \boldsymbol{w} = 0 \quad \textit{or} \quad \boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0 \quad \textit{for all } \boldsymbol{v} \textit{ in } \boldsymbol{V} \textit{ and all } \boldsymbol{w} \textit{ in } \boldsymbol{W}.$$

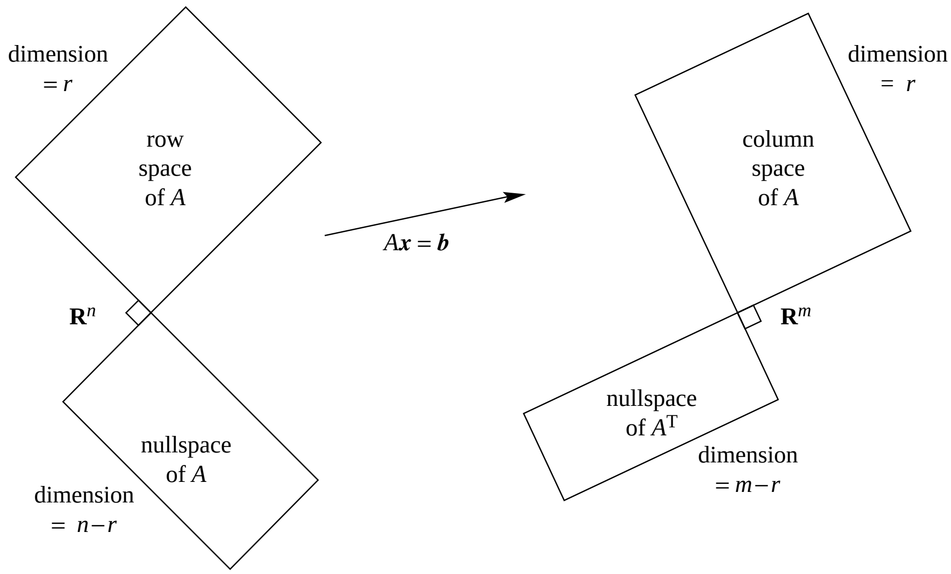

Figure 4.1 Two pairs of orthogonal subspaces. Dimensions add to n and add to m.

Example 1 The floor of your room (extended to infinity) is a subspace $\boldsymbol{V}$. The line where two walls meet is a subspace $\boldsymbol{W}$ (one-dimensional). Those subspaces are orthogonal. Every vector up the meeting line is perpendicular to every vector in the floor. The origin $(0, 0, 0)$ is in the corner. We assume you don't live in a tent.

Example 2 Suppose $\boldsymbol{V}$ is still the floor but $\boldsymbol{W}$ is a wall (a two-dimensional space). The wall and floor look like orthogonal subspaces but they are not! You can find vectors in $\boldsymbol{V}$ and $\boldsymbol{W}$ that are not perpendicular. In fact a vector running along the bottom of the wall is also in the floor. This vector is in both $\boldsymbol{V}$ and $\boldsymbol{W}$—and it is not perpendicular to itself.

When a vector is in two orthogonal subspaces, it *must* be zero. It is perpendicular to itself. It is $\boldsymbol{v}$ and it is $\boldsymbol{w}$, so $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v} = 0$. This has to be the zero vector.

The crucial examples for linear algebra come from the fundamental subspaces. Zero is the only point where the nullspace meets the row space. The spaces meet at $90°$.

4A Every vector $\boldsymbol{x}$ in the nullspace of A is perpendicular to every row of A, because $A\boldsymbol{x} = \mathbf{0}$. ***The nullspace and row space are orthogonal subspaces***.

To see why $\boldsymbol{x}$ is perpendicular to the rows, look at $A\boldsymbol{x} = \mathbf{0}$. Each row multiplies $\boldsymbol{x}$:

$$A\boldsymbol{x} = \begin{bmatrix} & \text{row } 1 & \\ & \vdots & \\ & \text{row } m & \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{x} \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{1}$$

The first equation says that row 1 is perpendicular to $\boldsymbol{x}$. The last equation says that row m is perpendicular to $\boldsymbol{x}$. *Every row has a zero dot product with* $\boldsymbol{x}$. Then $\boldsymbol{x}$ is perpendicular to every combination of the rows. The whole row space $\boldsymbol{C}(A^{\mathrm{T}})$ is orthogonal to the whole nullspace $\boldsymbol{N}(A)$.

Here is a second proof of that orthogonality for readers who like matrix shorthand. The vectors in the row space are combinations $A^{\mathrm{T}}\boldsymbol{y}$ of the rows. Take the dot product of $A^{\mathrm{T}}\boldsymbol{y}$ with any $\boldsymbol{x}$ in the nullspace. *These vectors are perpendicular*:

$$\boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{y}) = (A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}^{\mathrm{T}}\boldsymbol{y} = 0. \tag{2}$$

We like the first proof. You can see those rows of A multiplying $\boldsymbol{x}$ to produce zeros in equation (1). The second proof shows why A and A^{T} are both in the Fundamental Theorem. A^{T} goes with $\boldsymbol{y}$ and A goes with $\boldsymbol{x}$. At the end we used $A\boldsymbol{x} = \mathbf{0}$.

Example 3 The rows of A are perpendicular to $\boldsymbol{x} = (1, 1, -1)$ in the nullspace:

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 2 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{gives the dot products} \quad \begin{matrix} 1 + 3 - 4 = 0 \\ 5 + 2 - 7 = 0 \end{matrix}$$

Now we turn to the other two subspaces. In this example, the column space is all of $\mathbf{R}^2$. The nullspace of A^{T} is only the zero vector. Those two subspaces are also orthogonal.

4B Every vector $\boldsymbol{y}$ in the nullspace of A^{T} is perpendicular to every column of A. ***The left nullspace and the column space are orthogonal in*** $\mathbf{R}^m$.

Apply the original proof to A^{T}. Its nullspace is orthogonal to its row space—and the row space of A^{T} is the column space of A. Q.E.D.

For a visual proof, look at $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$. Each column of A multiplies $\boldsymbol{y}$ to give 0:

$$A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} (\text{column } 1)^{\mathrm{T}} \\ \cdots \\ (\text{column } n)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{y} \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix}. \tag{3}$$

The dot product of $\boldsymbol{y}$ with every column of A is zero. Then $\boldsymbol{y}$ in the left nullspace is perpendicular to each column—and to the whole column space.

Orthogonal Complements

Very Important The fundamental subspaces are more than just orthogonal (in pairs). Their dimensions are also right. Two lines could be perpendicular in $\mathbf{R}^3$, but they *could not be* the row space and nullspace of a 3 by 3 matrix. The lines have dimensions 1 and 1, adding to 2. The correct dimensions r and $n-r$ must add to $n=3$. The fundamental subspaces have dimensions 2 and 1, or 3 and 0. The subspaces are not only orthogonal, they are *orthogonal complements*.

DEFINITION The ***orthogonal complement*** of $\boldsymbol{V}$ contains *every* vector that is perpendicular to $\boldsymbol{V}$. This orthogonal subspace is denoted by $\boldsymbol{V}^{\perp}$ (pronounced "$\boldsymbol{V}$ perp").

By this definition, the nullspace is the orthogonal complement of the row space. *Every* $\boldsymbol{x}$ that is perpendicular to the rows satisfies $A\boldsymbol{x}=\mathbf{0}$.

The reverse is also true (automatically). If $\boldsymbol{v}$ is orthogonal to the nullspace, it must be in the row space. Otherwise we could add this $\boldsymbol{v}$ as an extra row of the matrix, without changing its nullspace. The row space would grow, which breaks the law $r+(n-r)=n$. We conclude that $\boldsymbol{N}(A)^{\perp}$ is exactly the row space $\boldsymbol{C}(A^{\mathrm{T}})$.

The left nullspace and column space are not only orthogonal in $\mathbf{R}^m$, they are also orthogonal complements. Their dimensions add to the full dimension m.

> ***Fundamental Theorem of Linear Algebra*, Part 2**
>
> ***The nullspace is the orthogonal complement of the row space (in $\mathbf{R}^n$).***
> ***The left nullspace is the orthogonal complement of the column space (in $\mathbf{R}^m$).***

Part 1 gave the dimensions of the subspaces. Part 2 gives the 90° angles between them. The point of "complements" is that every $\boldsymbol{x}$ can be split into a *row space component* $\boldsymbol{x}_r$ and a *nullspace component* $\boldsymbol{x}_n$. When A multiplies $\boldsymbol{x}=\boldsymbol{x}_r+\boldsymbol{x}_n$, Figure 4.2 shows what happens:

The nullspace component goes to zero: $A\boldsymbol{x}_n=\mathbf{0}$.

The row space component goes to the column space: $A\boldsymbol{x}_r=A\boldsymbol{x}$.

Every vector goes to the column space! Multiplying by A cannot do anything else. But more than that: *Every vector in the column space comes from one and only one vector $\boldsymbol{x}_r$ in the row space*. Proof: If $A\boldsymbol{x}_r=A\boldsymbol{x}'_r$, the difference $\boldsymbol{x}_r-\boldsymbol{x}'_r$ is in the nullspace. It is also in the row space, where $\boldsymbol{x}_r$ and $\boldsymbol{x}'_r$ came from. This difference must be the zero vector, because the spaces are perpendicular. Therefore $\boldsymbol{x}_r=\boldsymbol{x}'_r$.

There is an r by r invertible matrix hiding inside A, if we throw away the two nullspaces. From the row space to the column space, A is invertible. The "pseudoinverse" will invert it in Section 7.4.

Example 4 Every diagonal matrix has an r by r invertible submatrix:

$$A=\begin{bmatrix}3&0&0&0&0\\0&5&0&0&0\\0&0&0&0&0\end{bmatrix}\quad\text{contains}\quad\begin{bmatrix}3&0\\0&5\end{bmatrix}.$$

The rank is $r=2$. The other eleven zeros are responsible for the nullspaces.

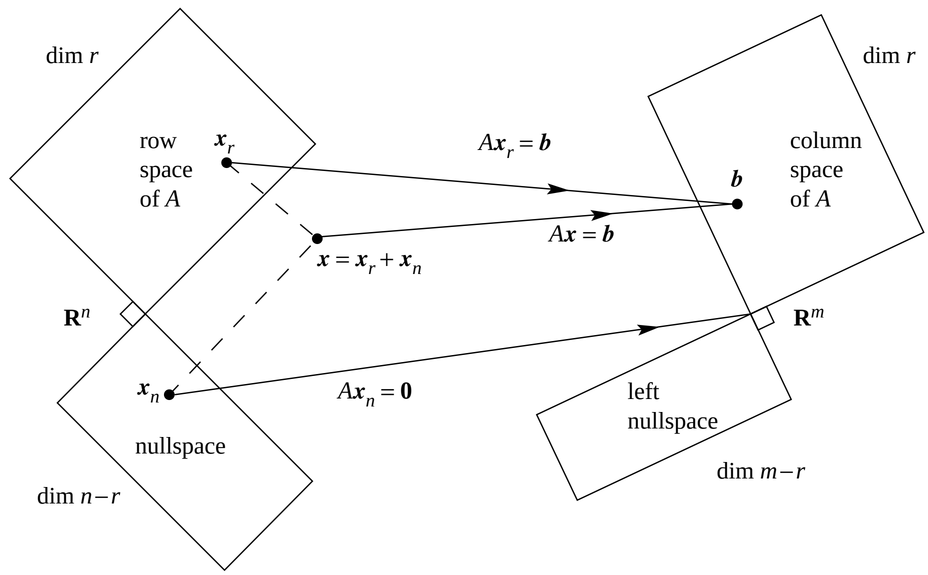

Figure 4.2 The true action of A times $\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$. Row space vector $\boldsymbol{x}_r$ to column space, nullspace vector $\boldsymbol{x}_n$ to zero.

Section 7.4 will show how every A becomes a diagonal matrix, when we choose the right bases for $\mathbf{R}^n$ and $\mathbf{R}^m$. This ***Singular Value Decomposition*** is a part of the theory that has become extremely important in applications.

Combining Bases from Subspaces

What follows are some valuable facts about bases. They were saved until now—when we are ready to use them. After a week you have a clearer sense of what a basis is (*independent* vectors that *span* the space). Normally we have to check both of these properties. When the count is right, one property implies the other:

4C Any n linearly independent vectors in $\mathbf{R}^n$ must span $\mathbf{R}^n$. They are a basis. Any n vectors that span $\mathbf{R}^n$ must be independent. They are a basis.

Starting with the correct number of vectors, one property of a basis produces the other. This is true in any vector space, but we care most about $\mathbf{R}^n$. When the vectors go into the columns of an n by n *square* matrix A, here are the same two facts:

4D If the n columns of A are independent, they span $\mathbf{R}^n$. So $A\boldsymbol{x} = \boldsymbol{b}$ is solvable. If the n columns span $\mathbf{R}^n$, they are independent. So $A\boldsymbol{x} = \boldsymbol{b}$ has only one solution.

Uniqueness implies existence and existence implies uniqueness. ***Then*** A ***is invertible***.

If there are no free variables (uniqueness), there must be n pivots. Then back substitution solves $A\boldsymbol{x} = \boldsymbol{b}$ (existence). Starting in the opposite direction, suppose $A\boldsymbol{x} = \boldsymbol{b}$ can always be solved (existence of solutions). Then elimination produced no zero rows. There are n pivots and no free variables. The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ (uniqueness of solutions).

With a basis for the row space and a basis for the nullspace, we have $r+(n-r) = n$ vectors—the right number. Those n vectors are independent.[2] *Therefore they span* $\mathbf{R}^n$. They are a basis:

Each $\boldsymbol{x}$ in $\mathbf{R}^n$ is the sum $\boldsymbol{x}_r + \boldsymbol{x}_n$ of a row space vector $\boldsymbol{x}_r$ and a nullspace vector $\boldsymbol{x}_n$.

This confirms the splitting in Figure 4.2. It is the key point of orthogonal complements—the dimensions add to n and all vectors are fully accounted for.

Example 5 For $A = \begin{bmatrix} I & I \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix}$, write any vector $\boldsymbol{x}$ as $\boldsymbol{x}_r + \boldsymbol{x}_n$.

$(1, 0, 1, 0)$ and $(0, 1, 0, 1)$ are a basis for the row space. $(1, 0, -1, 0)$ and $(0, 1, 0, -1)$ are a basis for the nullspace of A. Those four vectors are a basis for $\mathbf{R}^4$. Any $\boldsymbol{x} = (a, b, c, d)$ can be split into $\boldsymbol{x}_r$ in the row space and $\boldsymbol{x}_n$ in the nullspace:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \frac{a+c}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{b+d}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \frac{a-c}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \frac{b-d}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

■ REVIEW OF THE KEY IDEAS ■

1. Subspaces $\boldsymbol{V}$ and $\boldsymbol{W}$ are orthogonal if every $\boldsymbol{v}$ in $\boldsymbol{V}$ is orthogonal to every $\boldsymbol{w}$ in $\boldsymbol{W}$.

2. $\boldsymbol{V}$ and $\boldsymbol{W}$ are "orthogonal complements" if $\boldsymbol{W}$ contains **all** vectors perpendicular to $\boldsymbol{V}$ (and vice versa). Inside $\mathbf{R}^n$, the dimensions of $\boldsymbol{V}$ and $\boldsymbol{W}$ add to n.

3. The nullspace $\boldsymbol{N}(A)$ and the row space $\boldsymbol{C}(A^{\mathrm{T}})$ are orthogonal complements, from $A\boldsymbol{x} = \mathbf{0}$. Similarly $\boldsymbol{N}(A^{\mathrm{T}})$ and $\boldsymbol{C}(A)$ are orthogonal complements.

[2] If a combination of the vectors gives $\boldsymbol{x}_r + \boldsymbol{x}_n = \mathbf{0}$, then $\boldsymbol{x}_r = -\boldsymbol{x}_n$ is in both subspaces. It is orthogonal to itself and must be zero. All coefficients of the row space basis and nullspace basis must be zero—which proves independence of the n vectors together.

4. Any n independent vectors in $\mathbf{R}^n$ will span $\mathbf{R}^n$.

5. Every $\boldsymbol{x}$ in $\mathbf{R}^n$ has a nullspace component $\boldsymbol{x}_n$ and a row space component $\boldsymbol{x}_r$.

■ WORKED EXAMPLES ■

4.1 A Suppose $\boldsymbol{S}$ is a six-dimensional subspace of $\boldsymbol{R}^9$. What are the possible dimensions of subspaces orthogonal to $\boldsymbol{S}$? What are the possible dimensions of the orthogonal complement $\boldsymbol{S}^\perp$ of $\boldsymbol{S}$? What is the smallest possible size of a matrix A that has row space $\boldsymbol{S}$? What is the shape of its nullspace matrix N? How could you create a matrix B with extra rows but the same row space? Compare the nullspace matrix for B with the nullspace matrix for A.

Solution If $\boldsymbol{S}$ is six-dimensional in $\boldsymbol{R}^9$, subspaces orthogonal to $\boldsymbol{S}$ can have dimensions $0, 1, 2, 3$. The orthogonal complement $\boldsymbol{S}^\perp$ will be the largest orthogonal subspace, with dimension 3. The smallest matrix A must have 9 columns and 6 rows (its rows are a basis for the 6-dimensional row space $\boldsymbol{S}$). Its nullspace matrix N will be 9 by 3, since its columns contain a basis for $\boldsymbol{S}^\perp$.

If row 7 of B is a combination of the six rows of A, then B has the same row space as A. It also has the same nullspace matrix N. (The special solutions $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3$ will be the same. Elimination will change row 7 of B to all zeros.)

4.1 B The equation $x - 4y - 5z = 0$ describes a plane $\boldsymbol{P}$ in $\boldsymbol{R}^3$ (actually a subspace). The plane $\boldsymbol{P}$ is the nullspace $\boldsymbol{N}(A)$ of what 1 by 3 matrix A? Find a basis $\boldsymbol{s}_1, \boldsymbol{s}_2$ of special solutions of $x - 3y - 4z = 0$ (these would be the columns of the nullspace matrix N). Also find a basis for the line $\boldsymbol{P}^\perp$ that is perpendicular to $\boldsymbol{P}$. Then split $\boldsymbol{v} = (6, 4, 5)$ into its nullspace component $\boldsymbol{v}_n$ in $\boldsymbol{P}$ and its row space component $\boldsymbol{v}_r$ in $\boldsymbol{P}^\perp$.

Solution The equation $x - 3y - 4z = 0$ is $A\boldsymbol{x} = \boldsymbol{0}$ for the 1 by 3 matrix $A = [1 \ \ -3 \ \ -4]$. Columns 2 and 3 are free (no pivots). The special solutions with free variables "1 and 0" are $\boldsymbol{s}_1 = (3, 1, 0)$ and $\boldsymbol{s}_2 = (4, 0, 1)$. The row space of A (which is the line $\boldsymbol{P}^\perp$) certainly has basis $\boldsymbol{z} = (1, -3, -4)$. This is perpendicular to $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ and their plane $\boldsymbol{P}$.

To split $\boldsymbol{v}$ into $\boldsymbol{v}_n + \boldsymbol{v}_r = (c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2) + c_3\boldsymbol{z}$, solve for the numbers c_1, c_2, c_3:

$$\begin{bmatrix} 3 & 4 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix} \qquad \begin{array}{l} \text{leads to} \quad c_1 = 1, \ c_2 = 1, \ c_3 = -1 \\ \boldsymbol{v}_n = \boldsymbol{s}_1 + \boldsymbol{s}_2 = (7, 1, 1) \quad \text{is in} \quad \boldsymbol{P} = \boldsymbol{N}(A) \\ \boldsymbol{v}_r = -\boldsymbol{z} = (-1, 3, 4) \quad \text{is in} \quad \boldsymbol{P}^\perp = \boldsymbol{C}(A^{\mathrm{T}}). \end{array}$$

Problem Set 4.1

Questions 1–12 grow out of Figures 4.1 and 4.2.

1 Construct any 2 by 3 matrix of rank one. Copy Figure 4.1 and put one vector in each subspace (two in the nullspace). Which vectors are orthogonal?

2 Redraw Figure 4.2 for a 3 by 2 matrix of rank $r = 2$. Which subspace is $\boldsymbol{Z}$ (zero vector only)? The nullspace part of any vector $\boldsymbol{x}$ in $\mathbf{R}^2$ is $\boldsymbol{x}_n = \_\_\_\_$.

3 Construct a matrix with the required property or say why that is impossible:

(a) Column space contains $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$, nullspace contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(b) Row space contains $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$, nullspace contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(c) $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has a solution and $A^{\mathrm{T}}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(d) Every row is orthogonal to every column (A is not the zero matrix)

(e) Columns add up to a column of zeros, rows add to a row of 1's.

4 If $AB = 0$ then the columns of B are in the ____ of A. The rows of A are in the ____ of B. Why can't A and B be 3 by 3 matrices of rank 2?

5 (a) If $A\boldsymbol{x} = \boldsymbol{b}$ has a solution and $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$, then $\boldsymbol{y}$ is perpendicular to ____.

(b) If $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{c}$ has a solution and $A\boldsymbol{x} = \boldsymbol{0}$, then $\boldsymbol{x}$ is perpendicular to ____.

6 This is a system of equations $A\boldsymbol{x} = \boldsymbol{b}$ with *no solution*:

$$\begin{aligned} x + 2y + 2z &= 5 \\ 2x + 2y + 3z &= 5 \\ 3x + 4y + 5z &= 9 \end{aligned}$$

Find numbers y_1, y_2, y_3 to multiply the equations so they add to $0 = 1$. You have found a vector $\boldsymbol{y}$ in which subspace? Its dot product $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b}$ is 1.

7 Every system with no solution is like the one in Problem 6. There are numbers $y_1, \ldots, y_m$ that multiply the m equations so they add up to $0 = 1$. This is called

Fredholm's Alternative: **Exactly one of these problems has a solution**

$$A\boldsymbol{x} = \boldsymbol{b} \quad \textbf{OR} \quad A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0} \quad \text{with} \quad \boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 1.$$

If $\boldsymbol{b}$ is not in the column space of A, it is not orthogonal to the nullspace of A^{T}. Multiply the equations $x_1 - x_2 = 1$ and $x_2 - x_3 = 1$ and $x_1 - x_3 = 1$ by numbers y_1, y_2, y_3 chosen so that the equations add up to $0 = 1$.

8 In Figure 4.2, how do we know that $A\boldsymbol{x}_r$ is equal to $A\boldsymbol{x}$? How do we know that this vector is in the column space? If $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and $\boldsymbol{x} = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ what is $\boldsymbol{x}_r$?

9 If $A\boldsymbol{x}$ is in the nullspace of A^{T} then $A\boldsymbol{x} = \boldsymbol{0}$. Reason: $A\boldsymbol{x}$ is also in the _____ of A and the spaces are _____. *Conclusion*: $A^{\mathrm{T}}A$ *has the same nullspace as* A.

10 Suppose A is a symmetric matrix ($A^{\mathrm{T}} = A$).

(a) Why is its column space perpendicular to its nullspace?

(b) If $A\boldsymbol{x} = \boldsymbol{0}$ and $Az = 5z$, which subspaces contain these "eigenvectors" $\boldsymbol{x}$ and z? Symmetric matrices have perpendicular eigenvectors.

11 (Recommended) Draw Figure 4.2 to show each subspace correctly for

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}.$$

12 Find the pieces $\boldsymbol{x}_r$ and $\boldsymbol{x}_n$ and draw Figure 4.2 properly if

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Questions 13–23 are about orthogonal subspaces.

13 Put bases for the subspaces $\boldsymbol{V}$ and $\boldsymbol{W}$ into the columns of matrices V and W. Explain why the test for orthogonal subspaces can be written $V^{\mathrm{T}}W =$ zero matrix. This matches $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0$ for orthogonal vectors.

14 The floor $\boldsymbol{V}$ and the wall $\boldsymbol{W}$ are not orthogonal subspaces, because they share a nonzero vector (along the line where they meet). No planes $\boldsymbol{V}$ and $\boldsymbol{W}$ in $\mathbf{R}^3$ can be orthogonal! Find a vector in the column spaces of both matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 5 & 4 \\ 6 & 3 \\ 5 & 1 \end{bmatrix}$$

This will be a vector $A\boldsymbol{x}$ and also $B\widehat{\boldsymbol{x}}$. Think 3 by 4 with the matrix $[\,A \;\; B\,]$.

15 Extend problem 14 to a p-dimensional subspace $\boldsymbol{V}$ and a q-dimensional subspace $\boldsymbol{W}$ of $\mathbf{R}^n$. What inequality on $p+q$ guarantees that $\boldsymbol{V}$ intersects $\boldsymbol{W}$ in a nonzero vector? These subspaces cannot be orthogonal.

16 Prove that every $\boldsymbol{y}$ in $\boldsymbol{N}(A^{\mathrm{T}})$ is perpendicular to every $A\boldsymbol{x}$ in the column space, using the matrix shorthand of equation (2). Start from $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$.

17 If $\boldsymbol{S}$ is the subspace of $\mathbf{R}^3$ containing only the zero vector, what is $\boldsymbol{S}^{\perp}$? If $\boldsymbol{S}$ is spanned by $(1, 1, 1)$, what is $\boldsymbol{S}^{\perp}$? If $\boldsymbol{S}$ is spanned by $(2, 0, 0)$ and $(0, 0, 3)$, what is $\boldsymbol{S}^{\perp}$?

18 Suppose $\boldsymbol{S}$ only contains two vectors $(1, 5, 1)$ and $(2, 2, 2)$ (not a subspace). Then $\boldsymbol{S}^{\perp}$ is the nullspace of the matrix $A =$ ____ . $\boldsymbol{S}^{\perp}$ is a subspace even if $\boldsymbol{S}$ is not.

19 Suppose $\boldsymbol{L}$ is a one-dimensional subspace (a line) in $\mathbf{R}^3$. Its orthogonal complement $\boldsymbol{L}^{\perp}$ is the ____ perpendicular to $\boldsymbol{L}$. Then $(\boldsymbol{L}^{\perp})^{\perp}$ is a ____ perpendicular to $\boldsymbol{L}^{\perp}$. In fact $(\boldsymbol{L}^{\perp})^{\perp}$ is the same as ____ .

20 Suppose $\boldsymbol{V}$ is the whole space $\mathbf{R}^4$. Then $\boldsymbol{V}^{\perp}$ contains only the vector ____ . Then $(\boldsymbol{V}^{\perp})^{\perp}$ is ____ . So $(\boldsymbol{V}^{\perp})^{\perp}$ is the same as ____ .

21 Suppose $\boldsymbol{S}$ is spanned by the vectors $(1, 2, 2, 3)$ and $(1, 3, 3, 2)$. Find two vectors that span $\boldsymbol{S}^{\perp}$. This is the same as solving $A\boldsymbol{x} = \mathbf{0}$ for which A?

22 If $\boldsymbol{P}$ is the plane of vectors in $\mathbf{R}^4$ satisfying $x_1 + x_2 + x_3 + x_4 = 0$, write a basis for $\boldsymbol{P}^{\perp}$. Construct a matrix that has $\boldsymbol{P}$ as its nullspace.

23 If a subspace $\boldsymbol{S}$ is contained in a subspace $\boldsymbol{V}$, prove that $\boldsymbol{S}^{\perp}$ contains $\boldsymbol{V}^{\perp}$.

Questions 24–30 are about perpendicular columns and rows.

24 Suppose an n by n matrix is invertible: $AA^{-1} = I$. Then the first column of A^{-1} is orthogonal to the space spanned by which rows of A?

25 Find $A^{\mathrm{T}}A$ if the columns of A are unit vectors, all mutually perpendicular.

26 Construct a 3 by 3 matrix A with no zero entries whose columns are mutually perpendicular. Compute $A^{\mathrm{T}}A$. Why is it a diagonal matrix?

27 The lines $3x + y = b_1$ and $6x + 2y = b_2$ are ____ . They are the same line if ____ . In that case (b_1, b_2) is perpendicular to the vector ____ . The nullspace of the matrix is the line $3x + y =$ ____ . One particular vector in that nullspace is ____ .

28 Why is each of these statements false?

(a) $(1, 1, 1)$ is perpendicular to $(1, 1, -2)$ so the planes $x + y + z = 0$ and $x + y - 2z = 0$ are orthogonal subspaces.

(b) The subspace spanned by $(1, 1, 0, 0, 0)$ and $(0, 0, 0, 1, 1)$ is the orthogonal complement of the subspace spanned by $(1, -1, 0, 0, 0)$ and $(2, -2, 3, 4, -4)$.

(c) If two subspaces meet only in the zero vector, the subspaces are orthogonal.

29 Find a matrix with $\boldsymbol{v} = (1, 2, 3)$ in the row space and column space. Find another matrix with $\boldsymbol{v}$ in the nullspace and column space. Which pairs of subspaces can $\boldsymbol{v}$ *not* be in?

30 Suppose A is 3 by 4 and B is 4 by 5 and $AB = 0$. Prove $\operatorname{rank}(A) + \operatorname{rank}(B) \leq 4$.

31 The command $N = \mathsf{null}(A)$ will produce a basis for the nullspace of A. Then the command $B = \mathsf{null}(N')$ will produce a basis for the ____ of A.